



NORTH-HOLLAND

**The Generalized Spectral-Radius Theorem: An Analytic-Geometric Proof**

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**ABSTRACT**

Let  $\Sigma$  be a bounded set of complex matrices,  $\Sigma^m = \{A_1 \dots A_m : A_i \in \Sigma\}$ . The generalized spectral-radius theorem states that  $\rho(\Sigma) = \hat{\rho}(\Sigma)$ , where  $\rho(\Sigma)$  and  $\hat{\rho}(\Sigma)$  are defined as follows:  $\rho(\Sigma) = \limsup \rho_m(\Sigma)\{1/m\}$ , where  $\rho_m(\Sigma) = \sup \{\rho(A) : A \in \Sigma^m\}$  with  $\rho(A)$  the spectral radius;  $\hat{\rho}(\Sigma) = \limsup \hat{\rho}_m(\Sigma)\{1/m\}$ , where  $\hat{\rho}_m(\Sigma) = \sup \{\|A\| : A \in \Sigma^m\}$  with  $\|\cdot\|$  any matrix norm. We give an elementary proof, based on analytic and geometric tools, which is in some ways simpler than the first proof by Berger and Wang.

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**0. INTRODUCTION**

Let  $\Sigma$  denote a bounded set of complex  $k$ -by- $k$  matrices. For  $m \geq 1$ ,  $\Sigma^m$  is the set of all products of matrices in  $\Sigma$  of length  $m$ ,

$$\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma, i = 1, \dots, m\}.$$

Denoting by  $\rho(A)$  the spectral radius and by  $\|A\|$  an operator norm of a matrix  $A$ , one defines two different generalized spectral radii of  $\Sigma$  by the following expressions:

(1) The *generalized spectral radius*  $\rho(\Sigma)$  is

$$\rho(\Sigma) = \limsup_{m \rightarrow \infty} [\rho_m(\Sigma)]\{1/m\}, \quad (0.1)$$

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where

$$\rho_m(\Sigma) = \sup\{\rho(A) : A \in \Sigma^m\}.$$

(2) The *joint spectral radius*  $\hat{\rho}(\Sigma)$  is

$$\hat{\rho}(\Sigma) = \limsup_{m \rightarrow \infty} [\rho_m(\Sigma)]^{1/m}, \quad (0.2)$$

where

$$\hat{\rho}_m(\Sigma) = \sup\{\|A\| : A \in \Sigma^m\}.$$

We remark here that  $\hat{\rho}(\Sigma)$  is independent of the norm used, that  $\rho(\Sigma) \leq \hat{\rho}(\Sigma)$ , and that, as can be easily established from [3, (3.12)], the  $\limsup$  in (0.2) is actually a limit.

The generalized spectral-radius theorem is

**THEOREM 1.** *For any bounded set  $\Sigma$  of real or complex  $k$ -by- $k$  matrices one has*

$$\rho(\Sigma) = \hat{\rho}(\Sigma). \quad (0.3)$$

It is the aim of this note to provide a geometric-algebraic proof.

The joint spectral radius was introduced by Rota and Strang in [8], and the generalized spectral radius by Daubechies and Lagarias in [3]. The latter conjectured that Theorem 1 holds for finite sets. Theorem 1 was proved for real matrices by Berger and Wang [2]. By the remarks at the beginning of Section 3, this proves Theorem 1 also in the complex case. In their paper the authors use standard results from ring theory.

In the proof given below we apply mainly analytic and geometric tools. We feel that these are more appropriate and also somewhat more accessible to many readers. On one or two occasions we use elementary results of [2].

Having given the relevant definitions and stated the main result, we will in Section 1 prove Theorem 1 under some rather strict additional assumptions. This is stated as Lemma 3.

In Section 2 a suitable reduction for  $\Sigma$  is proved (Lemma 4). This leads immediately to a proof of Theorem 1 for a finite set  $\Sigma$ . Section 3 lifts this result to the case that  $\Sigma$  is bounded. Here we have to use some elementary facts about convex sets.

## 1. PRELIMINARIES

Let  $\Sigma^0 = \{I\}$  and  $M = \bigcup_{n=0}^{\infty} \Sigma^n$ . Following the terminology in [3], we call the set  $\Sigma$  product bounded if  $M$  is a bounded set. It is a standard result that for such  $\Sigma$  there exists a vector norm  $\|\cdot\|$  such that  $\|Ax\| \leq \|x\|$  for  $A \in \Sigma$ ,  $x \in \mathbb{C}^k$ . Simply choose  $\|x\| = \sup \{\|Ax\|_2 : A \in M\}$ , where  $\|\cdot\|_2$  denotes the Euclidean vector norm. Then  $\sup \{\|A\| : A \in \Sigma\} \leq 1$ , where  $\|A\|$  denotes the operator norm. Also  $\hat{\rho}(\Sigma) \leq 1$ . We will need:

LEMMA 1.  $\hat{\rho}(\Sigma) = \inf_{\nu \text{ an operator norm}} \sup\{\nu(A) : A \in \Sigma\}$ .

*Proof.* We remark first that this is a special case of the main result of [8], but in line with our goal to keep things as simple as possible, we give an elementary proof. Let  $\tilde{\rho} = \tilde{\rho}(\Sigma)$  denote the expression on the right-hand side. For  $\tau > \tilde{\rho}$  there exists an operator norm  $\nu$  such that  $\nu(A) \leq \tau$  for all  $A \in \Sigma$ . It follows that  $\hat{\rho} \leq \tau$  and hence  $\hat{\rho} \leq \tilde{\rho}$ . On the other hand, let  $\tau > \hat{\rho}$ . Then  $\hat{\rho}_m(\Sigma) \cdot \tau^{-m}$  is bounded for all  $m \in \mathbb{N}$ , i.e.,  $(1/\tau)\Sigma$  is product bounded, and hence  $\sup\{\|A\| : A \in (1/\tau)\Sigma\} \leq 1$  for a suitable operator norm. This gives  $\tilde{\rho} \leq \tau$ . It follows that  $\tilde{\rho} \leq \hat{\rho}$ . ■

We need subsequently the following basic

LEMMA 2. Let  $\|\cdot\|$  denote a vector norm on  $\mathbb{C}^k$  and its operator norm in the space of  $k$ -by- $k$  matrices. There exists a constant  $C$  depending on  $\|\cdot\|$  such that for any  $z \in \mathbb{C}^k$ ,  $\|z\| = 1$ , and any  $k$ -by- $k$  matrix  $A$  with  $\|A\| \leq 1$  and eigenvalues  $\lambda_1, \dots, \lambda_k$ , the inequality

$$\min_i |\lambda_i - \mu| \leq C \|Az - z\|^{1/k} \quad (1.1)$$

holds.

*Proof.* As all norms are equivalent, we need to prove (1.1) for the Euclidean vector norm only. Let  $\mu$  be an eigenvalue of some matrix  $B$ , and  $\sigma_1 \leq \dots \leq \sigma_k$  the singular values of  $A - \mu I$ . As  $B - \mu I$  is singular, we have  $\sigma_1 \leq \|\mu I - A - (\mu I - B)\| = \|A - B\|$  and  $\sigma_i \leq \|A\| + \|B\|$ , where  $\|\cdot\|$  is the spectral norm. Then  $\min |\lambda_i - \mu|^k \leq |\det(A - \mu I)| = \prod \sigma_i \leq \|A - B\|(\|A\| + \|B\|)^{k-1}$ . Hence

$$\min |\lambda_i - \mu| \leq (\|A\| + \|B\|)^{1-1/k} \|A - B\|^{1/k}. \quad (1.2)$$

(This reasoning closely follows [5]; see also [1]). Choose  $B = A + (z - Az)z^H$ , where  $z^H = \bar{z}'$ .

Then  $Bz = z$ ,  $\|A - B\| = \|z - Az\|$ . The inequality (1.2) for  $\mu = 1$  gives (1.1) for the Euclidean and the spectral norm, and hence Lemma 2 is proved. ■

We show now that (0.3) holds in a special case.

**LEMMA 3.** *Let  $\Sigma = \{A_1, \dots, A_m\}$  be finite, product bounded, and such that  $\hat{\rho}(\Sigma) = 1$ . Then  $\rho(\Sigma) = \hat{\rho}(\Sigma)$ .*

*Proof.* Choose an operator norm  $\|\cdot\|$  such that

$$\|A_i\| \leq 1 \quad \text{for } i = 1, \dots, m. \quad (1.3)$$

We claim that there exists a sequence  $\{d_i\}_{i=1}^\infty$ ,  $1 \leq d_i \leq m$ , such that  $\|T_n\| = 1$  for all  $n \in \mathbb{N}$ , where

$$T_n = A_{d_n} \cdots A_{d_1}.$$

For a finite sequence  $w = (n_1, \dots, n_s)$  of integers in  $\{1, \dots, m\}$  (a *word*), define  $A_w = A_{n_s} \cdots A_{n_1} \in \Sigma^s$  and  $W = \{w : \|A_w\| = 1\}$ . Then  $W$  is infinite, as otherwise  $\hat{\rho}(\Sigma) < 1$  follows. (See e.g. [3, (3.12)], where it is shown that  $\hat{\rho}(\Sigma) \leq \hat{\rho}_m(\Sigma)(1/m)$ .)

Now by a simple selection procedure one can find a sequence  $\{d_i\}_{i=1}^\infty$  such that for any  $n$  there are infinitely many words  $w \in W$ , the first  $n$  digits of which are  $d_1, \dots, d_n$ . This is an example of König's infinity lemma [6].

Choose a sequence  $\{x_n\}_{n=1}^\infty$  such that  $\|x_n\| = \|T_n x_n\| = 1$  for  $n \in \mathbb{N}$ . For an appropriate subsequence  $\{n_i\}_{i=1}^\infty$  we have

$$x_{n_i} \rightarrow \xi, \quad T_{n_i} \xi \rightarrow \eta \quad \text{as } i \rightarrow \infty.$$

From  $T_n \xi = T_n x_n + T_n(\xi - x_n)$  we get  $\|\eta\| = 1$ . Hence for any given  $\epsilon > 0$  we can choose  $r = n_i$ ,  $s = n_{i+1}$  such that

$$\|T_s \xi - T_r \xi\| \leq \epsilon, \quad \|T_r \xi\| \geq \frac{1}{2}. \quad (1.4)$$

Now  $T_s = AT_r$  for some  $A \in \Sigma^{s-r}$ . Choose  $z = T_r \xi / \|T_r \xi\|$ . Then (1.4) gives, as  $\|A\| \leq 1$ ,

$$\|Az - z\| \leq 2\epsilon.$$

By (1.1),  $A$  has an eigenvalue  $\lambda$  satisfying  $|\lambda - 1| = O(\epsilon^{1/k})$ . It follows that  $\rho(\Sigma) \geq 1$ , and hence  $\rho(\Sigma) = \hat{\rho}(\Sigma)$ . ■

## 2. FINITE $\Sigma$

In this section we prove Theorem 1 for finite  $\Sigma$  by reducing the general case to the case treated in Lemma 3. This reduction is done using

LEMMA 4. *Let  $\Sigma$  be a bounded set of complex  $k$ -by- $k$  matrices,  $\rho(\Sigma) = 1$ . If  $\Sigma$  is not product bounded, then there is a nonsingular  $S$  and  $1 \leq n_1 < k$  such that for all  $A \in \Sigma$*

$$S^{-1}AS = \begin{pmatrix} A_{(2)} & * \\ 0 & A_{(1)} \end{pmatrix}, \quad (2.1)$$

where  $A_{(1)}$  is  $n_1$ -by- $n_1$ .

*Proof.* By Lemma 1, for any  $\epsilon > 0$  there exists a vector norm  $\nu_\epsilon$  such that

$$\nu_\epsilon(Ax) \leq (1 + \epsilon)\nu_\epsilon(x) \quad \text{for } A \in \Sigma, \quad x \in \mathbb{C}^k, \quad (2.2)$$

which we normalize by

$$\max_{x \neq 0} \frac{\nu_\epsilon(x)}{\|x\|_2} = 1, \quad (2.3)$$

where, as before,  $\|\cdot\|_2$  is the Euclidean norm.

As the set of functions  $\{\nu_\epsilon\}_{\epsilon > 0}$  is bounded by (2.3) and equicontinuous on the compact set  $\{x : \|x\|_2 \leq 1\}$ , the theorem of Arzela and Ascoli shows that there exists a convergent subsequence

$$\nu_{\epsilon_i}(x) \rightarrow \nu(x), \quad \epsilon_i \rightarrow 0, \quad x \in \mathbb{C}^k. \quad (2.4)$$

The limit function  $\nu$  is a seminorm satisfying

$$\nu(Ax) \leq \nu(x), \quad A \in \Sigma, \quad x \in \mathbb{C}^k, \quad (2.5)$$

and by the normalization (2.3) we can show that  $\nu(x) \neq 0$ .

The nullspace  $V = \{x : \nu(x) = 0\}$  of the seminorm  $\nu$  is a linear substance. As  $\Sigma$  is not product bounded,  $V \neq \{0\}$ . By (2.5),  $V$  is an invariant subspace for all  $A \in \Sigma$ , and  $\dim V = n - n_1 > 0$ , so in a suitable basis all  $A \in \Sigma$  are reduced to the form (2.1). ■

Now we are able to prove Theorem 1 under the additional assumption that  $\Sigma$  is finite.

THEOREM 1'. *For a finite set  $\Sigma$  of  $k$ -by- $k$  matrices,*

$$\rho(\Sigma) = \hat{\rho}(\Sigma). \quad (0.3)$$

*Proof.* In the case that  $\hat{\rho}(\Sigma) = 0$ , (0.3) holds trivially, as  $\rho \leq \hat{\rho}$ . So we assume  $\hat{\rho}(\Sigma) = 1$ . If  $k = 1$ , then  $\hat{\rho} = \rho$  holds, as  $\rho(A)$  is a norm.

We proceed by induction. Either  $\Sigma$  is product bounded, whence by Lemma 3 we have  $\rho = \hat{\rho}$ , or we have the situation of Lemma 4. Defining  $\Sigma_{(i)} = \{A_{(i)} : A \in \Sigma\}$ ,  $i = 1, 2$ , where  $A_{(i)}$  is given in (2.1), we have

$$\rho(\Sigma) = \max\{\rho(\Sigma_{(i)}) : i = 1, 2\}$$

and

$$\hat{\rho}(\Sigma) = \max\{\hat{\rho}(\Sigma_{(i)}) : i = 1, 2\}$$

(See e.g. [2].) As the  $A_{(i)}$  have dimensions less than  $k$ , we have  $\hat{\rho}(\Sigma_{(i)}) = \rho(\Sigma_{(i)})$  and hence  $\rho(\Sigma) = \hat{\rho}(\Sigma)$ . ■

### 3. BOUNDED $\Sigma$

In this section we lift Theorem 1' to the case of a bounded set  $\Sigma$  of  $k$ -by- $k$  complex matrices. As we can view complex  $k$ -by- $k$  matrices as acting on the real  $2k$ -vectors, we may assume that  $\Sigma$  is real. Our Theorem 1 is then an easy consequence of

LEMMA 5. *If  $\Sigma$  is a bounded set of real  $k$ -by- $k$  matrices, then*

$$\hat{\rho}(\Sigma) = \sup\{\hat{\rho}(\tilde{\Sigma}) : \tilde{\Sigma} \subset \Sigma, \tilde{\Sigma} \text{ finite}\}. \quad (3.1)$$

*Proof.* We denote by  $\Sigma^C$  the convex hull of  $\Sigma \cup (-\Sigma)$  and obtain by elementary calculations

$$\hat{\rho}(\Sigma) = \hat{\rho}(\Sigma \cup (-\Sigma)) = \hat{\rho}(\Sigma^C). \quad (3.2)$$

We may also assume that  $0 \in \overset{\circ}{\Sigma}^C$ , the interior of  $\Sigma^C$ , if we restrict the space to the linear subspace spanned by  $\Sigma^C$ .

Observe the following. If  $\bar{\Sigma}$  denotes the closure of  $\Sigma$ , then  $(\bar{\Sigma})^C = \overline{(\Sigma^C)}$ , and for any  $\delta > 0$  we have  $(1 - \delta)\bar{\Sigma}^C \subset \Sigma^C \subset \overline{\Sigma^C}$ . Hence

$$(1 - \delta)\hat{\rho}(\bar{\Sigma}^C) \subset \hat{\rho}(\Sigma^C) \subset \hat{\rho}(\overline{\Sigma^C}),$$

and by (3.2) we infer  $\hat{\rho}(\Sigma) = \hat{\rho}(\bar{\Sigma})$ . Thus we may assume that  $\Sigma$  and hence  $\Sigma^C$  are closed.

By a result in [4] (Theorem 33), for a given  $\epsilon > 0$  there exists a convex polytope  $P$  such that

$$P \subset \Sigma^C \subset (1 + \epsilon)P. \quad (3.3)$$

(Let us remark here that this is not the formulation in [4]. According to Theorem 33 there exist polytopes  $P_1, P_2$  with Hausdorff distance  $\leq \epsilon$  such that  $P_1 \subset \Sigma^C \subset P_2$ . As the proof in [4] shows,  $P_1$  can be chosen to be a retraction by a factor  $(1 + k\epsilon)^{-1}$  with respect to an interior point  $\xi$ , where  $k$  depends on  $\Sigma^C$ . Choosing  $\xi = 0$ ,  $P = P_1$  and replacing  $k\epsilon$  by  $\epsilon$  gives (3.3).)

It follows that

$$\hat{\rho}(P) \leq \hat{\rho}(\Sigma) = \hat{\rho}(\Sigma^C) \leq (1 + \epsilon)\hat{\rho}(P). \quad (3.4)$$

$P$  is the convex hull of finitely many points of  $\Sigma^C$ , which again by Carathéodory's theorem are convex combinations of finitely many points  $\xi_1, \dots, \xi_p$  in  $\Sigma \cup -\Sigma$ . If  $P'$  is the convex hull of  $\{\xi_1, \dots, \xi_p\}$ , then

$$P \subset P' \subset \Sigma^C$$

and we have

$$\hat{\rho}(P) \leq \hat{\rho}(P') = \hat{\rho}(\{\xi_1, \dots, \xi_p\}) \leq \hat{\rho}(\Sigma). \quad (3.5)$$

Now let  $z_i = \pm \xi_i$ , where the sign is chosen so that  $z_i \in \Sigma$ , and  $\Sigma' = \{z_1, \dots, z_p\}$ . Then by (3.4) and (3.5)

$$\frac{1}{1 + \epsilon} \hat{\rho}(\Sigma) \leq \hat{\rho}(P) \leq \hat{\rho}(P') = \hat{\rho}(\Sigma') \leq \hat{\rho}(\Sigma),$$

and as  $\Sigma'$  is a finite subset of  $\Sigma$ , (2.1) follows. ■

The *proof of Theorem 1* is now easy. For given  $\epsilon > 0$

$$\begin{array}{ll} (1 - \epsilon) \hat{\rho}(\sigma) \leq \hat{\rho}(\Sigma') & \text{by Lemma 4 for a suitable finite } \Sigma' \subset \Sigma \\ = \rho(\Sigma') & \text{by Theorem 1'} \\ \leq \rho(\Sigma) & \text{by inclusion} \\ \leq \hat{\rho}(\Sigma) & \text{as remarked in the introduction.} \end{array}$$

Hence  $\hat{\rho}(\Sigma) = \rho(\Sigma)$ .

#### 4. CONCLUSION

We finish the paper by rephrasing the result from a different point of view. Let us assume that  $\Sigma = \{A_1, \dots, A_m\}$  is finite. Consider a sequence  $d = (d_1, d_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$ ; define  $T_n^d = A_{d_n} \cdots A_{d_1}$  and

$$\hat{\rho}(d) = \limsup_n \|T_n^d\|^{1/n}.$$

It is obvious that  $\hat{\rho}(d) \leq \hat{\rho}(\Sigma)$ . In [3], in the course of proving Theorem 3.1, it is shown that there exists  $d$  such that  $\|T_n^d\|^{1/n} \geq \hat{\rho}(\Sigma)$  for all  $n$ . Hence

$$\hat{\rho}(\Sigma) = \sup \{ \hat{\rho}(d) : d \in \{1, \dots, m\}^{\mathbb{N}} \}.$$

From  $\rho(A) = \lim \|A^m\|^{1/m}$  we infer that

$$\rho(A_{s_n} \cdots A_{s_1}) = \hat{\rho}(d),$$

where  $d$  is the sequence starting with  $s_1, \dots, s_n$  and continuing periodically. Hence

$$\rho(\Sigma) = \sup \{ \hat{\rho}(d) : d \in \{1, \dots, m\}^{\mathbb{N}}, \text{ periodic} \}.$$



So we can restate the result  $\rho(\Sigma) = \hat{\rho}(\Sigma)$  also in the form that the supremum of  $\hat{\rho}(d)$  over all sequences  $d$  is already reached by the periodic sequences. This should be considered in the light of the fact that  $\hat{\rho}(d)$  is in general not continuous if the metric in  $\{1, \dots, m\}^{\mathbb{N}}$  is chosen as  $\text{dist}(d, d')$

$$= \sum_{\nu=1}^{\infty} |d_{\nu} - d'_{\nu}| m^{-\nu}.$$

We have learned from the referee that Lagarias and Wang [7] have given special cases where for finite  $\Sigma$  the sup in (4.1) is attained.

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